

Question 1.

$$m = 100 \text{ g}$$

$$\text{extension } (e) = 10 \text{ cm}$$

$$\text{damping force} = 3500 \, dx/dt$$

$$\text{Spring constant } (k) = F/e$$

$$F = 0.1 \times 10 = 1 \text{ N}$$

$$e = 0.1 \text{ m}$$

$$k = \frac{1}{0.1} = 10 \text{ N/m}$$

damping force

$$F = \lambda \frac{dx}{dt} = 3500 \, dx/dt$$

$$\lambda = 3500$$

from Newton laws of motion

$$F = ma \quad \text{therefore}$$

$$k(x - L_0) + \lambda \frac{dx}{dt} = ma = -m \frac{d^2x}{dt^2}$$

$$(x - L_0) = 4 \text{ cm} = 0.04 \text{ m}$$

$$m \frac{d^2x}{dt^2} + \lambda \frac{dx}{dt} + (x - L_0) = 0$$

$$\frac{0.1}{10} \frac{d^2x}{dt^2} + \frac{3500}{10} \frac{dx}{dt} + 0.04 = 0 \quad \text{This is the go equation.}$$

$$0.01 \frac{d^2x}{dt^2} + 350 \frac{dx}{dt} + 0.04 = 0$$

Question 2

$$W(x) = \begin{cases} \frac{w_0}{c} (c-x) & 0 \leq x \leq c \\ 0 & c \leq x \leq 2c \end{cases}$$

$$M = -wx$$

Moments

$$M = EI \frac{d^2 y}{dx^2} = -wx$$

Integrating

$$EI \frac{dy}{dx} = -\frac{wx^2}{2} + C_1$$

at the support $\frac{dy}{dx} = 0$

$$EI(0) = 0$$

$$0 = -\frac{wx^2}{2} + C_1$$

$$C_1 = \frac{wc^2}{2}$$

$$EI \frac{dy}{dx} = -\frac{wx^2}{2} + \frac{wc^2}{2} \text{ slope for any point}$$

further integration

$$EI y = -\frac{wx^3}{6} + \frac{wc^2}{2} x + C_2$$

at $x=c$ $y=0$

$$EI(0) = 0$$

$$0 = -\frac{wc^3}{6} + \frac{wc^2}{2} c + C_2$$

$$C_2 = \frac{wc^3 - 3wc^3}{6}$$

$$k_2 = \frac{-2\omega c^3}{6}$$

$$k_2 = \frac{-\omega c^3}{3}$$

$$EIy = \frac{-\omega x^3}{6} + \frac{\omega c^2}{2}x - \frac{\omega c^3}{3}$$

$$EIy = \frac{-\omega x^3 + 3\omega c^2x - 2\omega c^3}{6}$$

$$y = \frac{-\omega x^3 + 3\omega c^2x - 2\omega c^3}{6EI}$$

Replacing ω with $\frac{\omega_0}{c}(c-x)$

$$y = \frac{\omega_0 x^3(c-x) + 3c^2x\omega_0(c-x) - 2c^3\omega_0(c-x)}{6cEI}$$

$$y = \frac{\omega_0 x^3(c-x) + 3c^2x\omega_0(c-x) - 2c^3\omega_0(c-x)}{6cEI}$$

Question 3

$$y_1' = -36y_1 - 6y_2$$

$$y_2' = 6y_1 - y_2$$

Writing in Matrix form

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -36 & -6 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The characteristic equation

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} -36 & -6 \\ 6 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -36-\lambda & -6 \\ 6 & -1-\lambda \end{bmatrix}$$

Finding the determinant

$$(-36-\lambda)(-1-\lambda) - (6 \times -6) = 0$$

$$36 + 36\lambda + \lambda + \lambda^2 + 36 = 0$$

$$\lambda^2 + 37\lambda + 72 = 0$$

Eigen value / roots of the characteristic Equation

$$\lambda_1 = -2.0607 \text{ 4 d.p.}$$

$$\lambda_2 = -34.9393 \text{ 4 d.p.}$$

Finding the eigen vectors

the general form of the solution

$$\vec{y}(t) = \vec{y}_1 e^{-2.0607t} + \vec{y}_2 e^{-34.9393t}$$

$\vec{y}_1$ and $\vec{y}_2$ being arbitrary constants

$$\text{for } \lambda = -2.0607$$

$$\begin{bmatrix} -33.9393 & -6 \\ 6 & -1-\lambda \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\begin{bmatrix} -33.9393 & -6 \\ 6 & 1.0607 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$-33.9393v_{11} - 6v_{12} = 0$$

$$-33.9393v_{11} = 6v_{12}$$

$$v_{12} = \frac{-33.9393v_{11}}{6} = -5.6566v_{11} \text{ (4dp)}$$

also

$$v_1 = C_1 \begin{bmatrix} 1 \\ -5.6566 \end{bmatrix}$$

$$\text{for } \lambda = -33.9393$$

$$\begin{bmatrix} -1.0607 & -6 \\ 6 & 33.9393 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$-1.0607v_{21} - 6v_{22} = 0$$

$$-1.0607v_{21} = 6v_{22}$$

$$v_{21} = -5.6566v_{22} \text{ (4dp)}$$

$$6v_{21} + 33.9393v_{22} = 0$$

$$6v_{21} = -33.9393v_{22}$$

$$v_{21} = -5.6566v_{22} \text{ (4dp)}$$

$$v_2 = C_2 \begin{bmatrix} 1 \\ -5.6566 \end{bmatrix}$$

Where v_1 and v_2 represent $\vec{v}$

The eigen values are

$$\begin{bmatrix} 1 \\ -5.6566 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ -5.6566 \end{bmatrix}$$

Normalizing them they become

$$\begin{bmatrix} 0.1741 \\ -0.9847 \end{bmatrix}$$

and

$$\begin{bmatrix} 0.1741 \\ -0.9847 \end{bmatrix}$$

The general equation is

$$\vec{Y}(t) = C_1 \begin{bmatrix} 1 \\ -5.6566 \end{bmatrix} e^{-2.0607t} + C_2 \begin{bmatrix} 1 \\ -5.6566 \end{bmatrix} e^{-34.9936t}$$

Question 4

$$y'' + 3y' + 2y = r(t) \quad r(t) = \begin{cases} 0 & 0 \leq t < 1 \\ 3t^2 + 7 & t \geq 1 \end{cases} \quad \begin{matrix} y(0) = 0 \\ y'(0) = 0 \end{matrix}$$

for $0 \leq t < 1$

Find Laplace transform of both sides.

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{0\}.$$

$$[s^2 Y(s) - s y(0) - y'(0)] + [3s Y(s) - 3 y(0)] + 2 Y(s) = 0$$

$$y'(0) = 0.$$

$$y(0) = 0.$$

$$(s^2 + 3s + 2) Y(s) = 0.$$

$$Y(s) = \frac{0}{s^2 + 3s + 2} = 0$$

$$\text{Therefore } y(t) = 0.$$

for $t > 1$

$$y'' + 3y' + 2y = 3t^2$$

finding the Laplace transform of both sides.

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{3t^2\}.$$

Laplace transform of $3t^2$

$$at^n = \frac{n!}{s^{n+1}} \quad \text{hence.}$$

$$\mathcal{L}\{3t^2\} = \frac{3 \times 2!}{s^{2+1}} = \frac{6}{s^3}$$

$$(s^2 + 3s + 2)Y(s) = \frac{6}{s^3}$$

$$Y(s) = \frac{6}{s^3(s^2 + 3s + 2)} = \frac{6}{s^3(s+1)(s+2)}$$

$$\frac{6}{s^3(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{(s+1)} + \frac{E}{(s+2)}$$

Finding a common denominator

$$6 = A(s^2(s+1)(s+2)) + B(s(s+1)(s+2)) + C((s+1)(s+2)) + D(s^3(s+2)) + E(s^3(s+1))$$

simplifying

$$6 = A(s^4 + 3s^3 + 2s^2) + B(s^3 + 3s^2 + 2s) + C(s^2 + 3s + 2) + D(s^4 + 2s^3) + E(s^4 + s^3)$$

$$s^4(A+D+E) = 0$$

$$C = \frac{6}{2} = 3$$

$$s^3(3A+B+2D+E) = 0$$

$$s^2(2A+3B+C) = 0$$

$$B = -4.5$$

$$s(2B+3C) = 0$$

$$A = 5.25$$

$$2C = 6$$

$$2B + 3(3) = 0$$

$$D = -6$$

$$2B = -9$$

$$E = 0.75$$

$$B = -4.5$$

$$2A + 3(-4.5) + 3 = 0$$

$$2A = 10.5$$

$$A = 5.25$$

$$3(5.25) + 3(-4.5) + 2D + E = 0$$

$$5.25 + D + E = 0$$

$$2D + E = -10.5$$

$$D + E = -5.25$$

$$D + D = -6$$

$$\frac{6}{s^3(s+1)(s+2)} = \frac{5.25}{s} + \frac{-4.5}{s^2} + \frac{3}{s^3} + \frac{-6}{s+1} + \frac{0.75}{s+2}$$

finding the inverse laplace transform

$$y(t) = 5.25 - 4.5t + 1.5t^2 - 6e^{-t} + 0.75e^{-2t}$$

Therefore the solution is

$$y(t) = 5.25 - 4.5t + 1.5t^2 - 6e^{-t} + 0.75e^{-2t}$$

$(s^3(s+1))$

$(s^3(s+1))$

Question 5

RLC

$$R = 100 \Omega$$

$$L = 1 \text{ H}$$

$$C = 1 \times 10^{-4} \text{ F}$$

$$E_0 = 90 \text{ V}$$

$$\text{Initial } I_0 = 0$$

$$Q_0 = 0 \quad \text{at } t = 0$$

$$I_{\text{total}} = \frac{V}{R} + \frac{1}{L} \int V dt + C \frac{dv}{dt}$$

Through an inductor:

$$V = L \frac{di}{dt}$$

for a capacitor

$$i = C \frac{dv}{dt}$$

for a resistor

$$V = IR$$

$$I = V/R$$

The total current

$$I_{\text{total}} = I_R + I_L + I_C$$

$$I_R = 90/100 = 0.9 \text{ A}$$

$$I_L = \int \frac{V}{L} dt = \int_0^1 90 dt = \left[90t \right]_0^1 = [90 - 0] = 90 \text{ A}$$

$$I_C = C \frac{dv}{dt} = 1 \times 10^{-4} \frac{d}{dt} (90) = 0$$

$$I_{\text{total}} = 90 + 0.9 = 90.9 \text{ A}$$

$$I_{\text{total}} = 90.9 \text{ A}$$